

Rotations in Algebra and Physics: An Expository Overview on Orthogonal and Unitary Groups

Damien Koon

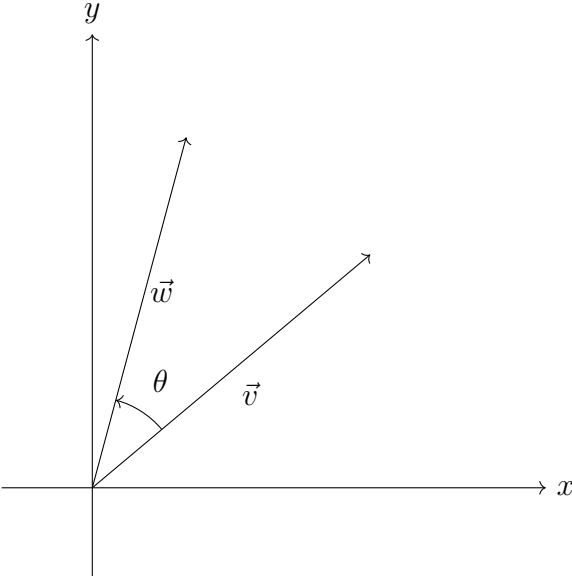
December 2025

Abstract

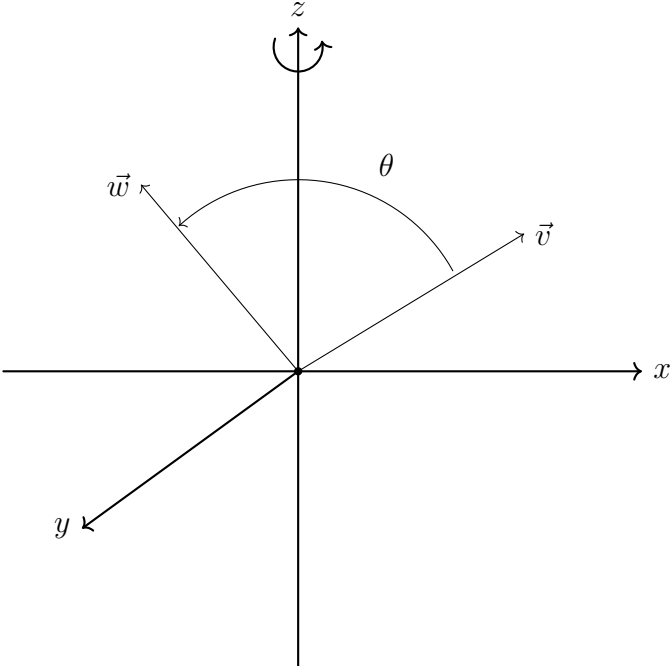
These notes present an expository overview of the orthogonal and unitary groups $O(n)$, $SO(n)$, $U(n)$, and $SU(n)$, focusing on $SU(2)$ and $SO(3)$. The content is based on an oral presentation given as a final examination for an abstract algebra course at Florida Tech in 2025 and has been expanded for clarity and completeness. We focus primarily on the mathematical motivation for these groups, with brief remarks on the role of spin in particle physics. No prior background in physics is assumed, but some background in abstract algebra is expected.

1 Introduction and Motivation

We begin by first asking the simple question, what is a rotation? Let's consider the standard Euclidean vector space, \mathbb{R}^2 . To rotate some vector $\vec{v} \in \mathbb{R}^2$, we only need to know what angle to rotate it by, say θ , to obtain our newly rotated vector $\vec{w} \in \mathbb{R}^2$.



Continuing on to rotations in \mathbb{R}^3 , now to rotate some vector $\vec{v} \in \mathbb{R}^3$, we again need an angle θ , but now we also need an axis to rotate about. For example, in the figure below we rotate about the z axis.



2 Rotations as Linear Transformations

Now we've seen rotations in 2 and 3 dimensions, but how can we generalize this to higher dimension?

Let's consider a vector $\vec{v} \in \mathbb{R}^n$. We want to know what operation will transform \vec{v} into our newly rotated vector $\vec{w} \in \mathbb{R}^n$.

In essence, we're looking for some function $R : \mathbb{R}^n \rightarrow \mathbb{R}^n$ where $R(\vec{v}) = \vec{w}$.

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \xrightarrow{\text{rotation}} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

To determine this operation, let's begin by analyzing the properties of rotation. We're going to focus on three main properties.

- 1) From the previous examples in 2 and 3 dimensions, we can see that rotations are linear transformations.
- 2) Rotations have the property that lengths and angles are preserved.
- 3) Rotations should preserve orientation to exclude other transformations such as reflections.

3 Orthogonal Groups

First, we will introduce the orthogonal groups, we will start by considering the vector space \mathbb{R}^n and seeing where the properties of rotation lead us.

3.1 The Groups $O(n)$ and $SO(n)$

Since rotations are linear, we know that for scalars a, b and vectors \vec{v}, \vec{w}

$$R(a\vec{v} + b\vec{w}) = aR(\vec{v}) + bR(\vec{w})$$

More importantly, this lets us know that a rotation R can be expressed with a matrix. Since R maps \mathbb{R}^n to itself, its matrix representation must be $n \times n$. Thus $R \in M_n(\mathbb{R})$.

We can mathematically represent the fact that rotations preserves lengths and angles by using the dot product.

$$\vec{v} \cdot \vec{w} = (R\vec{v}) \cdot (R\vec{w})$$

Since we've shown that rotations are linear, we can write the dot product as a matrix product and we observe that

$$\vec{v}^T \vec{w} = (R\vec{v})^T (R\vec{w}) = v^T (R^T R) \vec{w} \implies R^T R = I$$

We define matrices with these properties with the following group:

$$O(n) = \{R \in M_n(\mathbb{R}) : R^T R = I\}$$

We call this the orthogonal group, however rotations are a subset of this group.

We found that matrices satisfying $R^T R = I$ preserve lengths and angles. We now take the determinant of both sides:

$$\det(R^T R) = \det(I) = 1$$

However since $\det(R^T) = \det(R)$, we obtain

$$\det(R)^2 = 1 \implies \det(R) = \pm 1$$

Matrices with determinant +1 preserve orientation, while matrices with determinant -1 reverse orientation, as in reflections. Therefore $\det(R) = 1$.

We thus define the Special Orthogonal group:

$$SO(n) = \{R \in O(n) : \det(R) = 1\}$$

This group represents all of the rotations for \mathbb{R}^n . Precisely stated, $SO(n)$ is the group of linear isometries of \mathbb{R}^n that preserve orientation.

3.2 Basic Properties and Examples

We start by showing the basic example of a reflection in $O(2)$ which isn't in $SO(2)$

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It can be easily shown that $R^T R = I$, but $\det(R) = -1$.

We will now focus on $SO(3)$ since it tends to be the most physically relevant.

Without proof, we have that for a rotation $R \in SO(3)$ by an angle ϕ about the z-axis as is shown in Figure 2 in Section 1 has the form:

$$R = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We also have the general properties that $SO(3)$ is 3-dimensional and that every element of $SO(3)$ corresponds to a rotation about some axis by some angle. This is Euler's rotation theorem and is a key property for $SO(3)$ connecting algebra back to geometry.

4 Unitary Groups

Now that we have discussed the vector space \mathbb{R}^n , let's now consider \mathbb{C}^n .

Many of the same properties hold, but we will need to make some slight modifications.

4.1 The Groups $U(n)$ and $SU(n)$

Previously we have our rotation operator denoted by R , we will switch to U for "unitary."

We still have that rotations are linear, all that changes that instead of $R \in M_n(\mathbb{R})$, we have $U \in M_n(\mathbb{C})$.

Since rotations still preserve lengths and angle, we'll follow the same dot product argument as before, however instead of the transpose, we will use a "transpose-conjugate," more frequently called the Hermitian adjoint.

We denoted this operation with the dagger: $U^\dagger = U^{T*} = U^{*T}$. Following the same dot product argument as in Section 3.1 We obtain: $U^\dagger U = I$

Now we define the unitary group:

$$U(n) = \{U \in M_n(\mathbb{C}) : U^\dagger U = I\}$$

And as before, we require the determinant to be $+1$, however for a different reason than before. In \mathbb{C}^n , we don't have the same geometric notion of orientation as we do in \mathbb{R}^n . Instead, the determinant condition is to remove the phase factor of $U(n)$.

$$SU(n) = \{U \in U(n) : \det(U) = 1\}$$

4.2 Basic Properties and Examples

We will focus on $SU(2)$ throughout this section and we can represent $SU(2)$ entirely with matrices of the form:

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

We can check this with some quick examples. If we let $\alpha = 1, \beta = 0$ we obtain the identity, but less trivially if we let $\alpha = 1/\sqrt{2}, \beta = 1/\sqrt{2}i$, we obtain:

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}}i & +\frac{1}{\sqrt{2}} \end{pmatrix} \implies \det(U) = \frac{1}{2} - i^2 \frac{1}{2} = 1$$

We can see an interesting property if we let $\alpha = a + bi$ and $\beta = c + di$ where $a, b, c, d \in \mathbb{R}$. Then $|\alpha|^2 + |\beta|^2 = 1 \implies a^2 + b^2 + c^2 + d^2 = 1$ which is exactly the equation for the 3-sphere \mathbb{S}^3 .

Let's now consider Hamilton's quaternions.

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1\}$$

Take $q \in \mathbb{H}$ with unit length in \mathbb{R}^4 . That is $|q|^2 = 1$. Then we can see that $|a + bi + cj + dk|^2 = a^2 + b^2 + c^2 + d^2 = 1$ which is the same equation we saw previously. This means that $SU(2)$ is isomorphic to the group of unit quaternions.

Thus $SU(2)$ can be equivalently represented by \mathbb{S}^3 and the unit quaternions.

5 Relation to Lie Groups

We've now defined the four rotation groups, $O(n), SO(n), U(n), SU(n)$. Notably, in addition to each of these sets being groups, they are also smooth manifolds and Lie groups.

While further discussion of this is outside of the scope of this note, it does bridge the subject from the realm of algebra into differential geometry.

6 The Double Cover $SU(2) \rightarrow SO(3)$

A natural mapping from $SU(2)$ to $SO(3)$ is a two-to-one surjective homomorphism. A more concise way to say this is that $SU(2)$ is a double cover of $SO(3)$. We can see this by the first isomorphism theorem.

Recall that if $\phi : G \rightarrow H$ is a group homomorphism, then $G/\ker(\phi) \cong \text{Im}(\phi)$

$$\begin{aligned} \phi : SU(2) &\rightarrow SO(3) \\ SU(2)/\ker(\phi) &\cong SO(3) \end{aligned}$$

Where $\ker(\phi)$ is equal to the center of $SU(2)$ which is $\{\pm I\}$. Proving this will be left as an exercise to the reader. From this we can see that $SU(2)/\{\pm I\} \cong SO(3)$. But what does $SU(2)/\{\pm I\}$ represent?

Geometrically, $SU(2)$ is the 3-sphere, \mathbb{S}^3 as shown in Section 4.2. We can then identify 2 points on \mathbb{S}^3 to produce a space that is diffeomorphic to $SO(3)$. Thus every rotation corresponds to 2 opposite points on \mathbb{S}^3 .

Recall that any $R \in SO(3)$ has the form:

$$R = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Each rotation $R \in SO(3)$ corresponds to exactly two elements $\pm U \in SU(2)$. This is the explicit realization of the two-to-one nature of the double cover.

In more precise language, a vector $\vec{v} \in \mathbb{R}^3$ may be identified with a traceless Hermitian 2×2 matrix, say V . Then $U \in SU(2)$ acts on V by conjugation: $V \mapsto UVU^\dagger$

Additionally, both U and $-U$ produce the same transformation, since $(-U)V(-U)^\dagger = UVU^\dagger$

Thus each rotation in $SO(3)$ arises from two elements of $SU(2)$.

6.1 Physical Motivation: Spin

Elementary matter particles such as the electron are "spin-1/2 Fermions" and are represented with mathematical objects called "spinors."

A 360° rotation changes the sign of a spinor, while a 720° rotation returns it to its original state. This is a very strange realization! It appears that matter particles require a 4π rotation rather than the classically expected 2π rotation.

Spinors transform under $SU(2)$ and not $SO(3)$. The double cover as mentioned in Section 6 explains why a 2π rotation gives a minus sign. This is not a classical spatial rotation effect, but rather a property of quantum states.

This phenomenon is physically observed in neutron interferometry and not merely a book-keeping device. It can be difficult to visualize, but there are excellent representations such as Dirac's belt trick as a topological illustration.

7 Concluding Remarks

We then conclude that rotations in \mathbb{R}^n naturally lead to the group $SO(n)$, and that extending this to complex vector spaces leads to $SU(n)$.

We have also shown that $SU(2)$ and $SO(3)$ are closely related via a double cover and that this algebraic relation explains the physical behavior of spin in particle physics.

As has hopefully been made clear from this exposition, the seemingly simple questions "What is spin?" and "What does it mean to rotate?" are a very complicated matter that must be treated with great precision.

Bibliography

This exposition contains many standard results across various textbooks, internet resources, and was also informed by the following publicly available lecture series:

- *Spinors for Beginners* (YouTube playlist):
https://www.youtube.com/playlist?list=PLJHszsWbB6ho0o_wMb0b6T44KM_ABZtBs
- *Lie Groups, Lie Algebras, and Brackets* (YouTube playlist):
https://www.youtube.com/playlist?list=PLDcSwjT2BF_WDki-WvmJ__QOnLIHuNPbP

Acknowledgements

I would like to thank Dr. Sinha Babu for guidance throughout the Abstract Algebra course and for fostering an environment that encouraged mathematical curiosity. I am also grateful to my fellow peers and classmates in Abstract Algebra for helpful discussions and shared insight over the semester. Finally, I would like to thank Krupa Pothiwala for patiently sitting through multiple practice versions of this lecture and for provided invaluable feedback.